

Estimation of Bit and Frame Error Rates of Low-Density Parity-Check Codes on Binary Symmetric Channels

Abstract

A method for estimating the performance of low-density parity-check (LDPC) codes decoded by hard-decision iterative decoding algorithms on binary symmetric channels (BSC) is proposed. Based on the enumeration of the smallest weight error patterns that cannot be all corrected by the decoder, this method estimates both the frame error rate (FER) and the bit error rate (BER) of a given LDPC code with very good precision for all crossover probabilities of practical interest. Through a number of examples, we show that the proposed method can be effectively applied to both regular and irregular LDPC codes and to a variety of hard-decision iterative decoding algorithms. Compared with the conventional Monte Carlo simulation, the proposed method has a much smaller computational complexity, particularly for lower error rates.

Index Terms

Low-density parity-check (LDPC) codes, finite length LDPC codes, iterative decoding, hard-decision decoding algorithms, binary symmetric channels (BSC), error floor.

I. INTRODUCTION

Low-density parity-check (LDPC) codes, which were first proposed by Gallager [1], have attracted considerable attention over the past decade due to their capacity-achieving error performance and low complexity of the associated iterative decoding algorithms. Iterative decoding algorithms are particularly simple if binary messages are used. Such algorithms, which are referred to as hard-decision iterative algorithms, are the subject of this paper. Examples are the so-called Gallager algorithms A (GA) and B (GB) [1], [2], [3], their variants [4] and majority-based (MB) algorithms [5].

The asymptotic performance analysis of LDPC codes under iterative decoding algorithms is now a rather mature subject. On the other hand, the finite-length analysis of LDPC codes is still an active area of research with many open problems. In this paper, our focus is on practical LDPC codes with finite block lengths in the range of a few hundreds to a few thousands of bits.

In [6], Di *et al.* studied the average performance of the ensembles of regular finite-length LDPC codes over binary erasure channels (BEC), by relating the decoding failures of the belief propagation algorithm over a BEC to graphical objects called *stopping sets*. Another step in analyzing the performance of finite-length LDPC codes was taken by Richardson [7], where the performance of a *given* LDPC code, particularly in the error floor region, was related to the *trapping sets* of the underlying graph. Most recently, Cole *et al.* [8], devised an efficient algorithm for estimating the FER floors of soft-decision LDPC decoders on the AWGN channel using importance sampling. Estimations of the FER of GA and GB algorithms for certain categories of LDPC codes were also presented in [9], [10]. In an earlier paper [11], we derived upper and lower bounds on the FER of LDPC codes under hard-decision algorithms over a BSC. The bounds which were tight for sufficiently small crossover probabilities of the channel provided good estimates for the FER in this region.

In this paper, we propose a method to estimate the performance of LDPC codes decoded by hard-decision iterative algorithms over a BSC. Compared to the methods of [7], [8], [9], [10] which first identify the most relevant trapping sets and then evaluate their contribution to the error floor, our approach does not require the identification of the trapping sets and is thus much simpler. Moreover, our approach is not only applicable to the error floor region, but also provides very accurate estimates in the waterfall region. In addition, we provide estimates for the BER, which is not even discussed in [7], [8], [9], [10]. In fact one of the important contributions of this work is to shed some light on the generally complex structure of the error events for iterative decoding, and to be able to establish simple approximate relationships between these complex events and the smallest initial error patterns that the decoder fails to correct. These relationships are the key in deriving the estimates.

The rest of the paper is organized as follows. In section II, we discuss different decoder failures for hard-decision algorithms. We then propose our FER and BER estimation methods and discuss their complexity in Section III. Simulation results are given in Section IV. Section V concludes the paper.

II. FAILURES OF HARD-DECISION ITERATIVE DECODING ALGORITHMS

Hard-decision iterative decoding algorithms are initiated by the outputs from the BSC. Hard messages are then passed between the variable nodes and the check nodes in the Tanner graph of the code through the edges of the graph in both directions and iteratively. We assume that the alphabet space for messages is $\{+1, -1\}$ and that the decoder is symmetric [2]. To evaluate the performance, we can then assume that the all-one codeword is transmitted.

An important category of hard-decision iterative algorithms are majority-based (MB^ω) algorithms, where ω is referred to as the order of the algorithm. For an MB^ω algorithm, in iteration zero, check nodes are inactive and variable nodes pass their channel messages. In iteration $l \geq 1$, check nodes pass the product of their extrinsic incoming messages in iteration $l-1$, and variable nodes pass their channel messages unless at least one-half plus ω of their extrinsic incoming messages in iteration l disagree with the corresponding channel messages, in which case the negative of those (channel) messages are passed. In other words, for a variable node with degree d_v and channel message m_0 , $-m_0$ is passed if at least $\lceil d_v/2 \rceil + \omega$ of extrinsic incoming messages are $-m_0$, where $\lceil x \rceil$ is the smallest integer which is greater than or equal to x . The order ω can be any non-negative integer less than or equal to $d_v - 1 - \lceil d_v/2 \rceil$. This maximum value corresponds to GA.

Decoder failures of iterative algorithms can be related to graphical objects, called trapping sets [7]. A trapping set is defined as a set of variable nodes that cannot be eventually corrected by the decoder. In general, trapping sets depend not only on the structure of the graph but also on the decoder inputs (channel) and the decoding algorithm. For example, while for the belief propagation algorithm over the BEC, the trapping sets are precisely the stopping sets, for the maximum likelihood decoding over the BEC or any other channel, the trapping sets are the nonzero codewords. The structure of the trapping sets of hard-decision iterative decoding algorithms on a BSC is in general unknown. The following theorem identifies the structure of certain trapping sets for MB algorithms. This theorem is in fact a generalization of Facts 3 and 4 of [7]. Facts 3 and 4 of [7] only apply to GA and GB on regular Tanner graphs with variable node degree 3 and check node degree 6. The following theorem extends this to irregular codes with arbitrary degree distributions where different nodes are allowed to perform different majority-based algorithms.

Theorem 1: For an LDPC code with degree distribution pair (λ, ρ) , suppose that variable node j performs MB of order ω_j denoted by MB^{ω_j} ($0 \leq \omega_j \leq d_j - 1 - \lceil d_j/2 \rceil$), where d_j is the degree of variable node j . Consider a set of variable nodes S and denote its induced subgraph in the code's Tanner graph by $G(S)$. Also denote the check nodes which have an odd degree in $G(S)$ by $C_o(S)$. If each variable node j in the Tanner graph has at most $\lceil d_j/2 \rceil + \omega_j - 1$ neighbors from $C_o(S)$, then S is a trapping set.

Proof: We show that if all the variable nodes in S are initially in error and all the other nodes are initially correct, then the decoder fails to correct the errors and the initial error pattern remains unchanged throughout the iterations.

In the first iteration, the outgoing messages of a variable node j is the initial message $m_0 \in \{-1, 1\}$ of node j . Based on the updating rule for MB algorithms in the check nodes, it is easy to see that node j receives $-m_0$ along all the edges connected to $C_o(S)$ and m_0 along all the edges connected to $\overline{C_o(S)} = C \setminus C_o(S)$, where C is the set of all the check nodes in the Tanner graph. This is regardless of whether node j is in the set S or not. Due to the structure of S , node j will therefore have at most $\lceil d_j/2 \rceil + \omega_j - 1$ incoming messages with their value equal to $-m_0$. This implies that for each outgoing message of node j at the start of the second iteration, the number of extrinsic incoming messages with value $-m_0$ is also at most $\lceil d_j/2 \rceil + \omega_j - 1$. Hence the value of the outgoing message will remain m_0 . This is because, for the outgoing message to change, MB^{ω_j} requires at least $\lceil d_j/2 \rceil + \omega_j$ extrinsic incoming messages with value $-m_0$. This means that all the messages exchanged between variable nodes and check nodes, remain unchanged throughout the iterations. This implies that the decoded values of the variable nodes remain the same as the initial messages regardless of the number of iterations. ■

Theorem 1 describes an instance of one type of trapping set which we refer to as *fixed-pattern*. For hard-decision algorithms in general, however, this is not the only type of trapping sets. In our experiments with different LDPC codes and different hard-decision iterative decoding algorithms, we have observed the following types of decoder failures corresponding to different types of trapping sets. The observations are made by tracking the error positions at the output of the decoder throughout iterations.

- 1) Fixed-pattern: After a finite number of iterations, the error positions at the output of the decoder remain unchanged.
- 2) Oscillatory-pattern: After a finite number of iterations, the error positions at the output of

the decoder oscillate periodically within a small set of variable nodes.

- 3) Random-like: Error positions change with iterations in a seemingly random fashion. The errors seem to propagate in the Tanner graph and result in a larger number of errors at the output of the decoder even if the initial error pattern has only a small weight.

Note that in the first two cases, there could be a transition phase, through which the error pattern settles into the steady-state. Also noteworthy is that in all the random-like cases that we have observed, the trapping set includes all the bits in the codeword, i.e., no bit can be eventually corrected by the decoder. In our experiments, we also observe that the percentage of random-like failures out of the total number of failures increases on average as the weight of the initial error patterns increases.

The relationship between the trapping sets and the structure of the Tanner graph is in general complex. With the exception of Theorem 1 and code-specific results such as those of [9], [10], we are not aware of any other such result. Our experiments show that in many cases cycles in the Tanner graph are part of the trapping sets. The relationships among the cycles, the trapping sets and the initial error patterns that trap the decoder in the trapping sets, however, seem to be complex. For example, in a Tanner graph with girth 6, while variable nodes in cycles of length 6 can form trapping sets for GA, they may be corrected by other MB algorithms. Moreover, there are cases where the variable nodes in a cycle of length 6 form a fixed-pattern trapping set for GA if these nodes are initially in error with all the other nodes received correctly. The same variable nodes, however, can be corrected by GA if the initial error pattern also contains other variable nodes.

Due to the complexity of identifying and enumerating the different types of trapping sets for a given LDPC code under a given hard-decision iterative decoding algorithm, we take an approach different than that of [7], [8], [9], [10]. Our approach is universal in that it can be applied to any LDPC code with arbitrary degree distributions and to any decoding algorithm as long as the complexity of implementation is manageable. The algorithm is simply based on enumerating the initial error patterns of smallest weight that cannot be all corrected by the decoder. By using this information, we then estimate the contribution of all the other initial error patterns with larger weights to the total FER and BER.

III. ERROR RATE ESTIMATION

A. Frame Error Rate Estimation

Consider a given LDPC code with block length n decoded by a given hard-decision iterative algorithm over a BSC with crossover probability ε . Denote the set of all the error patterns of weight i by S_i , and those that cannot be corrected by the decoder by E_i . Clearly, $|S_i| = \binom{n}{i}$. Suppose that the decoder can correct all the error patterns of weight $J - 1$ and smaller, i.e., $|E_i| = 0, \forall i < J$. Also suppose that there are $|E_J| \neq 0$ weight- J error patterns that the decoder fails to correct. The FER is then equal to

$$FER = \sum_{i=J}^n P(e|i)p_i = \sum_{i=J}^n \frac{|E_i|}{|S_i|} p_i = \sum_{i=J}^n |E_i| \varepsilon^i (1 - \varepsilon)^{n-i}, \quad (1)$$

where e is the event of having a frame (codeword) decoded erroneously, i denotes the weight of the initial error pattern at the input of the decoder, and p_i is the probability of having i errors at the output of the channel (or the input of the decoder) given by the binomial distribution $p_i = \binom{n}{i} \varepsilon^i (1 - \varepsilon)^{n-i}$.

The first term of the summation in (1) is denoted by $P(J)$ and is equal to $|E_J| \varepsilon^J (1 - \varepsilon)^{n-J}$. To estimate the FER, we enumerate E_J and calculate $P(J)$ precisely. For the other terms in (1), we estimate $|E_i|$ as a function of $|E_J|$ as follows.

For a given $i > J$, we partition the set S_i as $S_i = S'_i \cup \overline{S'_i}$, where S'_i is the set of error patterns of weight i , each containing at least one element of E_J as its sub-pattern, and $\overline{S'_i}$ is the complement set of S'_i in S_i . The set $\overline{S'_i}$ thus contains the elements of S_i that do not have any elements of E_J as their sub-patterns. We make the following assumptions:

Assumption (a): No error pattern in the set S'_i can be corrected by the decoder.

Assumption (b): Every error pattern in the set $\overline{S'_i}$ can be corrected by the decoder.

By the above assumptions, we have $|E_i \cap S'_i| = |S'_i|$ and $|E_i \cap \overline{S'_i}| = 0$, and therefore $|E_i| = |E_i \cap S'_i| + |E_i \cap \overline{S'_i}| = |S'_i|$. The key in estimating $|E_i|$, $i > J$, is that $|S'_i|$ can be approximated as a function of $|E_J|$ using the following combinatorial arguments. Consider the probability \mathcal{P} that a randomly selected weight- i error pattern contains at least one element of E_J as its sub-pattern.

We then have

$$\mathcal{P} = \frac{|S'_i|}{|S_i|} \approx 1 - \left(1 - \frac{|E_J|}{|S_J|}\right)^{\binom{i}{j}}. \quad (2)$$

The first equality follows from the fact that there are $|S_i|$ equally likely possibilities out of which $|S'_i|$ are the favorable cases. The second part of the equation is a consequence of approximating the random experiment by a sequence of $\binom{i}{J}$ independent and identically distributed Bernoulli trials, each involving the selection of a weight- J error pattern with “success” defined as the pattern being in E_J . The probability of success is then $|E_J|/|S_J|$, and the probability of having at least one success in $\binom{i}{J}$ trials is $1 - (1 - |E_J|/|S_J|)^{\binom{i}{J}}$ following the binomial distribution. From (2), we have

$$|S'_i| \approx |S_i| \left[1 - \left(1 - \frac{|E_J|}{|S_J|} \right)^{\binom{i}{J}} \right] \approx |S_i| \frac{|E_J| \binom{i}{J}}{|S_J|} = |E_J| \binom{n-J}{i-J}, \quad (3)$$

where the second approximation is obtained by considering only the first two terms in the binomial expansion.

Note that Assumption (a) is valid for the belief propagation decoder on a BEC. For hard-decision iterative decoding algorithms on a BSC, although Assumptions (a) and (b) are not necessarily correct, they are approximately valid, especially for smaller values of i . In particular, our studies show that these assumptions are statistically viable and result in good approximations for $|E_i|$, $i > J$.

Combining (1) and (3) with $|E_i| \approx |S'_i|$, $i > J$, we derive the following estimates for the FER: “Lower” estimate,

$$FER_L(N) = P(J) + \sum_{i=J+1}^N |E_J| \binom{n-J}{i-J} \varepsilon^i (1-\varepsilon)^{n-i}, \quad (4)$$

“Upper” estimate,

$$FER_U(N) = P(J) + \sum_{i=J+1}^N |E_J| \binom{n-J}{i-J} \varepsilon^i (1-\varepsilon)^{n-i} + \left[1 - \sum_{i=0}^N \binom{n}{i} \varepsilon^i (1-\varepsilon)^{n-i} \right], \quad (5)$$

where $N \in \{J+1, J+2, \dots, n\}$ is a parameter to be selected for the best accuracy of the estimates. It is clear that $FER_L(N) \leq FER_U(N)$, and the equality holds if and only if $N = n$. The difference between the two estimates is the probability that the input error patterns to the decoder have a weight larger than N . While $FER_L(N)$ is derived based on the assumption that error patterns of weight larger than N occur with negligible probability, $FER_U(N)$ is obtained based on the assumption that for such input error patterns the decoder fails with probability

one. Our observations show that in practice there exists a certain threshold N_0 for error weights, around which a relatively abrupt change in the percentage of failures occurs. This is such that for error weights larger than N_0 , the probability of failure goes to one very rapidly.

Later in Section IV, we will see that $FER_U(N_0)$ provides a very accurate estimate of the FER for all channel crossover probabilities of interest. This suggests the following practical approach for determining N_0 . We perform Monte Carlo simulations at high FER values, say around $0.01 - 0.1$. We then choose N_0 such that the estimate $FER_U(N_0)$ is the closest to the simulated FER. As the Monte Carlo simulations are performed at high FER values, their complexity is low and easily manageable.

B. Bit Error Rate Estimation

At the first glance, the problem of BER estimation may seem very complicated as we observe that even for the initial error patterns of the same weight, depending on the type of the trapping set, the number of bits in error at the end of the decoding can be quite different. Our further study into this however reveals that despite this variety of possibilities, one can estimate the average number of bit errors depending on the initial weight i of the error patterns. In general, we expect the conditional average BER given the initial error weight i to be an increasing function of i . To simplify the analysis however, we partition the range of $J \leq i \leq n$ into two subsets, $J \leq i < N_0$ and $N_0 \leq i \leq n$. For these partitions, we estimate the average number of bit errors by J and M , respectively, where M is the estimate of the average number of bit errors for error patterns of weight N_0 , obtained by Monte Carlo simulations. We thus derive our estimate for the BER based on the FER estimate $FER_U(N_0)$, proposed in the previous subsection, as follows.

$$\begin{aligned} BER \approx & \frac{J}{n} P(J) + \frac{J}{n} \sum_{i=J+1}^{N_0-1} |E_J| \binom{n-J}{i-J} \varepsilon^i (1-\varepsilon)^{n-i} + \frac{M}{n} |E_J| \binom{n-J}{N_0-J} \epsilon^{N_0} (1-\epsilon)^{n-N_0} \\ & + \frac{M}{n} \left[1 - \sum_{i=0}^{N_0} \binom{n}{i} \varepsilon^i (1-\varepsilon)^{n-i} \right]. \end{aligned} \quad (6)$$

In the following we provide the rationale behind the derivation of (6). We recall that $|E_i| \approx |S'_i|$. We partition S'_i further as $S'_i = S''_i \cup \overline{S''}_i$, where S''_i is the set of error patterns of weight i that have one and only one element of the set E_J as their sub-pattern. Using similar discussions

as those used for the derivation of (3), we have

$$|S_i''| \approx |S_i| \binom{i}{J} \left(1 - \frac{|E_J|}{|S_J|}\right)^{(i)-1} \frac{|E_J|}{|S_J|}. \quad (7)$$

It appears that for the error patterns of weight $J \leq i < N_0$, the ratio of $|S_i''|/|\overline{S_i''}|$ is a large number and thus a large majority of the error patterns in the set E_i belong to the set S_i'' . This means that, for this range of error weights, many of the error patterns that result in decoding failures have one and only one element of the set E_J as their sub-pattern. For these error patterns we make the following assumption:

Assumption (c): For the error patterns of weight $J \leq i < N_0$, the number of bit errors at the end of decoding is approximately J .

Furthermore, we assume:

Assumption (d): For the error patterns of weight $N_0 \leq i \leq n$, the number of bit errors at the end of decoding is on average M , where M is the estimate of the average number of bit errors for error patterns of weight N_0 .

To determine M , we simulate a given number, say $10^5 - 10^6$, of randomly generated error patterns of weight N_0 .

Our experiments show that both Assumptions (c) and (d) are statistically viable and provide very good estimates of the total BER.

C. Computational Complexity

To obtain an estimate of FER, one needs to enumerate $\sum_{i=1}^J \binom{n}{i}$ error patterns. For different error patterns, the iterative decoder performs the decoding and counts the number of decoding failures. Suppose that we are interested in estimating the performance of the LDPC code at a FER of p using Monte Carlo simulations and would like to observe at least m codeword errors for a reliable result. Assuming that the average number of computations required for iterative decoding in the two cases are the same, the ratio of the computational complexities of the Monte Carlo simulation and the proposed estimation method is:

$$\eta = m/p \sum_{i=1}^J \binom{n}{i}. \quad (8)$$

For BER estimation, we need to perform an extra number of iterative decodings to estimate M , and about $100m$ iterative decoding to obtain N_0 . These are usually negligible compared to $\sum_{i=1}^J \binom{n}{i}$.

It can be seen that η is a function of n , J , p and m . It increases with increasing m and with decreasing p , n and J . It often appears that J is a small number (≤ 4). For example, we have tested all the codes with rate $1/2$ and $n \leq 2048$ in [12] under GA, and they all have $J \leq 4$. The value of m is often selected in the range of a few tens to a few hundreds. With given values for J , m and n , the proposed estimation method is more efficient than the Monte Carlo simulation ($\eta > 1$) if $p < m/\sum_{i=1}^J \binom{n}{i}$. In fact, for the block length and ε values of interest, the computational complexity of the proposed method can be much smaller than that of the Monte Carlo simulations.

One should also note that the value of η in (8) only reflects the saving in complexity compared to *one* Monte Carlo simulation point. Unlike our estimations that can be easily calculated for different values of ε once we obtain $|E_J|$, in Monte Carlo simulations, for each simulation point, a new set of input vectors has to be generated and simulated. This makes the proposed method even more attractive from the complexity viewpoint.

IV. SIMULATION RESULTS

To show that our method can be applied to both regular and irregular LDPC codes and to a variety of hard-decision iterative decoding algorithms, in this section, we perform experiments on four pairs of code/decoding algorithm. Code 1 is a $(200, 100)$ irregular LDPC code. The degree distributions for this code, which are optimized for the BSC and GA, are given by $\lambda(x) = 0.1115x^2 + 0.8885x^3$ and $\rho(x) = 0.26x^6 + 0.74x^7$ [3]. Code 2 is a $(210, 35)$ regular LDPC code which has a variable node degree 5 and check node degree 6. For the degree distribution of this code, MB algorithm of order zero (MB^0) has a better threshold than GA does [5]. Code 3 is a $(1008, 504)$ regular LDPC code with variable node degree 3 and check node degree 6 taken from [12]. Code 4, which is a $(1998, 1776)$ regular LDPC code taken from [12], has variable node degree 4 and check node degree 36. Tanner graphs for all the codes are free of cycles of length 4. Except Code 2, which is decoded by MB^0 , all the codes are decoded by GA. In our simulations, the maximum number of iterations is 100 for all the decoders and for each crossover probability, we simulate until we obtain 100 codeword errors.

Table I shows the values of J and $|E_J|$ for the four codes. It also shows the number and the percentages of different types of decoding failures for E_J . As can be seen, for all the codes, most of the decoding failures caused by initial error weight J have fixed patterns and there is no random-like failure.

Figures 1 - 4 show the FER and BER performances of Codes 1 - 4, obtained by simulations, respectively. In these figures, we have also given the upper and the lower estimates of the FER for different values of N , as well as the estimates for the BER. For the estimates, we have used $N_0 = 9, 13, 38$ and 10 , for Codes 1 - 4, respectively. The corresponding values of M are $7.73, 46.95, 143.93$ and 133.57 , respectively. From the figures, it can be seen that for each code, the lower estimate $FER_L(N)$ improves by increasing N . However, even the best estimate $FER_L(n)$ is only good at sufficiently small values of crossover probability. It fails to provide an accurate estimate at higher error rates. On the other hand, for all the codes, $FER_U(N_0)$ provides an impressively accurate estimate of the FER over the whole range of crossover probabilities of practical interest. It can also be seen that the BER estimates for all the codes follow the simulated BER curves very closely.

To compare the computational complexity of our proposed method with that of the conventional Monte Carlo simulations, we consider the following example.

Example 1: We consider estimating the performance of Code 1 at $p = 10^{-7}$. In this case, $J = 3$, $n = 200$ and we assume $m = 100$ as is the case for the simulation results presented in Fig. 1. With these values, we have $\eta = 750$. If we were to add another simulation point at $p = 10^{-8}$, the complexity of Monte Carlo simulations would increase to 8250 times that of our proposed method. For the larger block length of 1008, our proposed method is more efficient than Monte Carlo simulations if the target FER $p < 6 \times 10^{-7}$.

One should note that, as we discussed in Section III-C, for any given n and J , there exists a FER p' , for which our method is more efficient than the Monte Carlo simulations in estimating the FER over the range $p < p'$.

V. CONCLUSIONS

In this paper, we propose a method to estimate the error rate performance of LDPC codes decoded by hard-decision iterative decoding algorithms over a BSC. By only enumerating the smallest weight (J) error patterns that cannot be all corrected by the decoder, the proposed

method estimates both FER and BER for a given LDPC code over the whole crossover probability region of interest with very good accuracy. The proposed method is universal in that it is applicable in principle to both regular and irregular LDPC codes of arbitrary degree distributions and to any hard-decision iterative algorithm. This universality is partly due to the fact that, unlike previous approaches, our method is not based on identifying the trapping sets and their relationship with the Tanner graph structure of the code and the decoding algorithm.

Although the complexity of our proposed method is much less than that of the Monte Carlo simulations for many cases of interest, it still increases exponentially with J , essentially as n^J , where n is the block length. As a direction for future research, it would be interesting to find (to estimate) the number of error patterns of weight J that cannot be corrected by the decoder, using methods other than direct enumeration.

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TABLE I
CATEGORIZATION OF ERROR PATTERNS WITH WEIGHT J FOR DIFFERENT LDPC CODES.

	Code 1	Code 2	Code 3	Code 4
J	3	4	3	3
$ E_J $	1434	18	192	200479
Undetected Error	0	0	0	0
Fixed-Pattern	1303(90.87%)	14(77.8%)	165(85.9%)	200438(99.98%)
Oscillatory-Pattern	131(9.13%)	4(22.2%)	27(14.1%)	41(0.02%)
Random-like	0	0	0	0

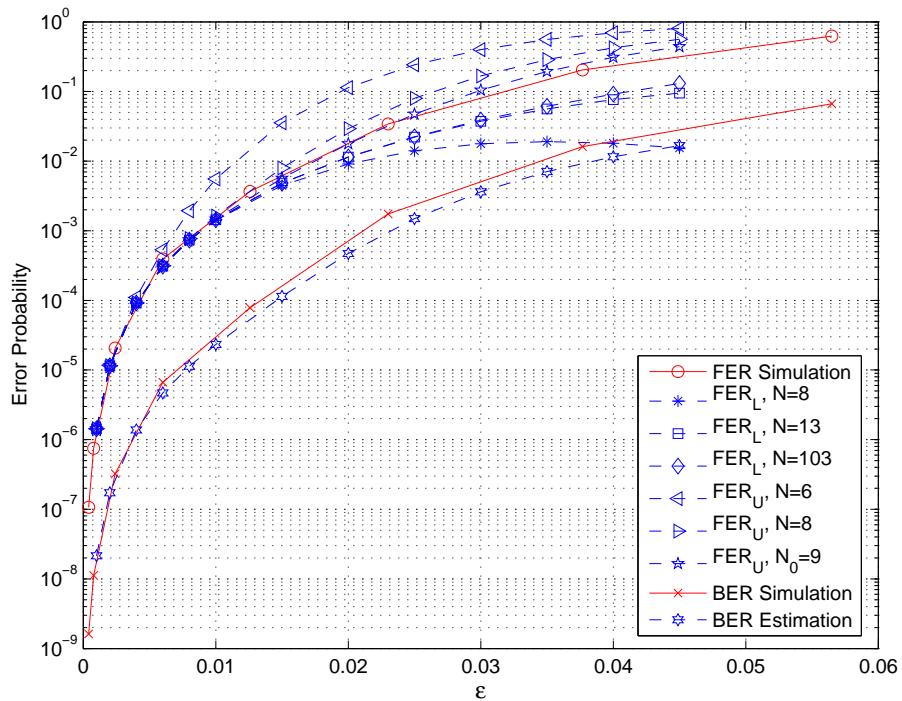


Fig. 1. FER and BER estimations and simulations of the $(200, 100)$ irregular code (Code 1, $\lambda(x) = 0.1115x^2 + 0.8885x^3$, $\rho(x) = 0.26x^6 + 0.74x^7$) decoded by GA.

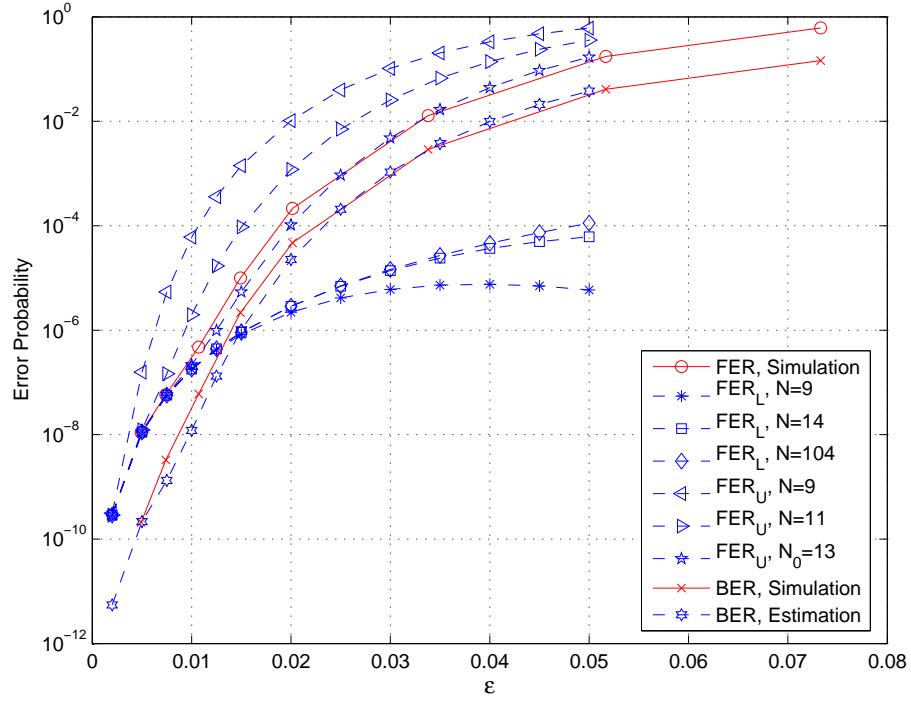


Fig. 2. FER and BER estimations and simulations of the (210, 35) regular code (Code 2, $\lambda(x) = x^4$, $\rho(x) = x^5$) decoded by MB⁰.

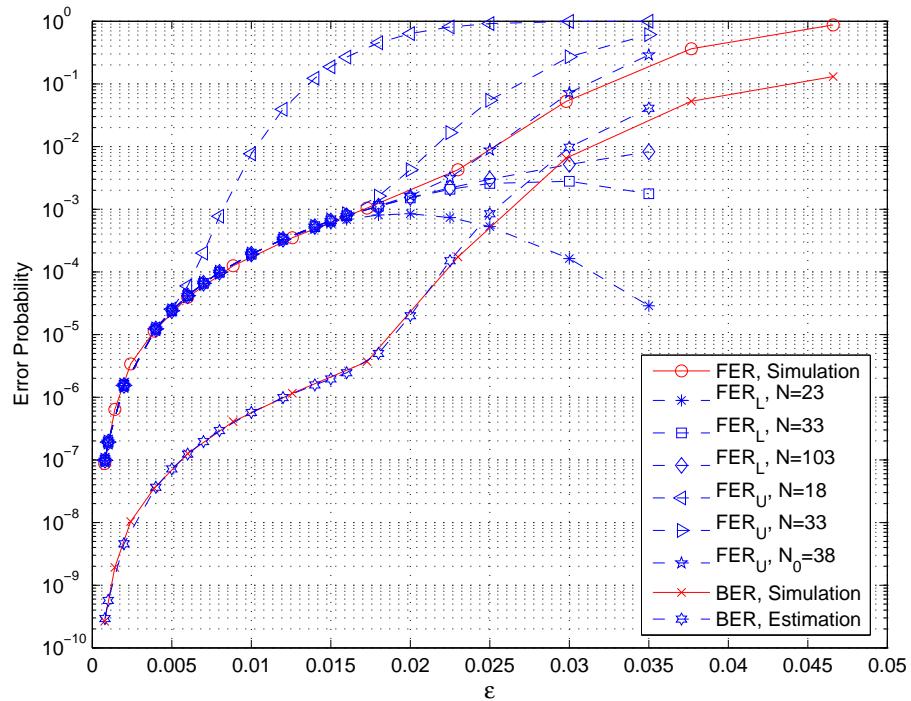


Fig. 3. FER and BER estimations and simulations of the (1008, 504) regular code (Code 3, $\lambda(x) = x^2$, $\rho(x) = x^5$) decoded by GA.

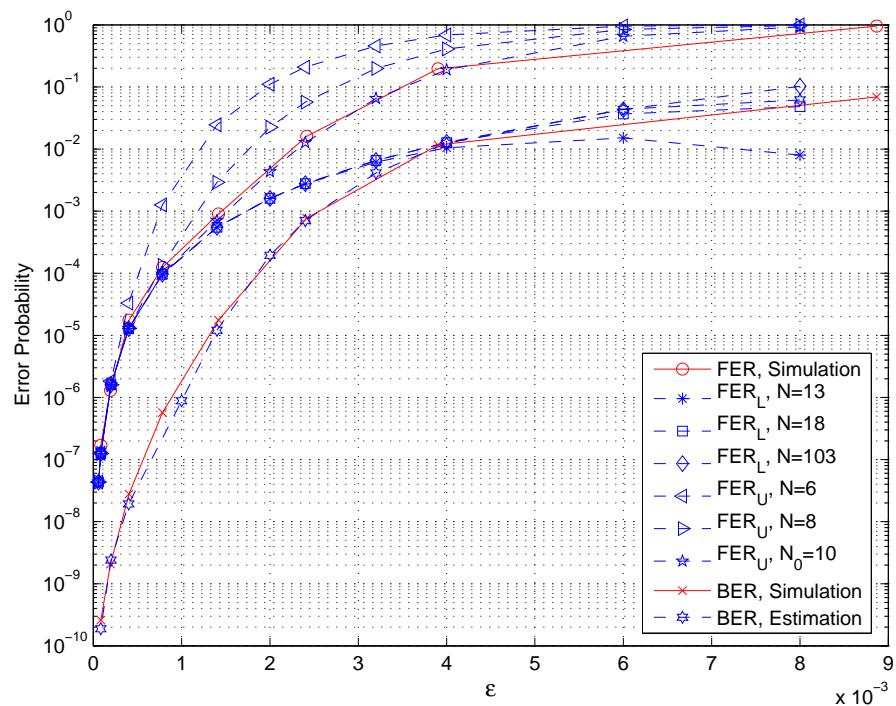


Fig. 4. FER and BER estimations and simulations of the (1998, 1776) regular code (Code 4, $\lambda(x) = x^3$, $\rho(x) = x^{35}$) decoded by GA.